JOURNAL OF APPROXIMATION THEORY 39, 89–92 (1983)

Degenerate Bernstein Polynomials

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Communicated by E. W. Cheney

Received August 5, 1982

For $f \in C[0, 1]$, the nth Bernstein polynomial is defined by

$$B_n(f; x) = B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}. \tag{1}$$

Generally, $B_n(x)$ is a polynomial of exact degree n, although degeneracies can occur. For example, if f itself is a polynomial of degree m, then $B_n(f; x)$ is also of degree m for all $n \geqslant m$ (although not equal to $f(x)$ except in the case $m = 1$). This result follows easily from an alternate form of the Bernstein polynomials [1, p. 13], namely,

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} \Delta_h^k f(0)\, x^k. \tag{2}$$

Here $h = 1/n$ and $\Delta_h^k f(0)$ is the kth forward difference of f,

$$\Delta_h^k f(0) = \Delta(\Delta_h^{k-1} f(0)) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f((k-j)h). \tag{3}$$

If $f(x)$ is a polynomial of degree m, then $\Delta_h^k f(0) = 0$ for all $n > m$, so that $B_n(f; x)$ is also of degree m. In this note we present a class of functions which have a sequence of degenerate Bernstein polynomials, and show, in addition, that surprising equalities occur for certain of these polynomials.

THEOREM. *Let $f(x)$ be a piecewise linear function with at most $m-1$ changes of slope, which can occur only at the points i/m, $i = 1, 2, \ldots, m-1$. Then for all natural numbers m, $B_{mn+1}(f; x)$ is of degree mn and, moreover, $B_{mn+1}(f; x) = B_{mn}(f; x)$.*

* Supported in part by a Temple University Grant-in-Aid of Research.

0021-9045/83 $3.00

Proof. By (2), $B_{mn}(x)=\sum_{k=0}^{mn} a_k x^k$, where $a_k=\binom{mn}{k}\Delta_h^k f(0)$, $h=1/mn$, and $B_{mn+1}(x)=\sum_{k=0}^{mn+1} b_k x^k$, where $b_k=\binom{mn+1}{k}\Delta_h^k f(0)$, $h=1/(mn+1)$. We will show that $b_k=a_k$, $k=0,1,\ldots,mn$, and $b_{mn+1}=0$.

Case 1. Suppose $(l-1)n+1<k\leqslant ln$, for some $l=1,2,\ldots,m$. Now $\Delta_h^k f(0)=\Delta(\Delta_h^{k-1}f(0))=\Delta_h^{k-1}f(h)-\Delta_h^{k-1}f(0)=\Delta(\Delta_h^{k-2}f(h))-\Delta(\Delta_h^{k-2}f(0))$. Applying the technique iteratively, we obtain $\Delta_h^k f(0)=\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\Delta_h^2 f((k-2-j)h)$, where $\Delta_h^2 f(x)=f(x+h)-2f(x+h)+f(x)$. In our case $h=1/mn$, so that

$$\Delta_h^k f(0)=\sum_{j=0}^{k-2}(-1)^j\binom{k-2}{j}\Delta_h^2 f\left(\frac{k-2-j}{mn}\right). \tag{4}$$

Let's recall now that f is a piecewise linear function, and that the second difference of a *linear* function is zero. Thus, the only terms in (4) that are different from zero are those in which the points $(k-2-j)/mn$, $(k-1-j)/mn$ and $(k-j)/mn$ "straddle" one of the points i/m at which $f(x)$ changes slope. Hence,

$$\Delta_h^k f(0)=\sum_{i=1}^{l-1}(-1)^{k-in-1}\binom{k-2}{k-in-1}\Delta_h^2 f\left(\frac{in-1}{mn}\right). \tag{5}$$

Now

$$\begin{aligned}\Delta_h^2 f\left(\frac{in-1}{mn}\right)&=f\left(\frac{in+1}{mn}\right)-2f\left(\frac{in}{mn}\right)+f\left(\frac{in-1}{mn}\right)\\&=\left[f\left(\frac{in+1}{mn}\right)-f\left(\frac{in}{mn}\right)\right]-\left[f\left(\frac{in}{mn}\right)-f\left(\frac{in-1}{mn}\right)\right]\\&=(S_{i+1}-S_i)/mn,\end{aligned}$$

where S_i is the slope of $f(x)$ on the interval $[(i-1)/m, i/m]$. Thus,

$$a_k=\binom{mn}{k}\sum_{i=1}^{l-1}(-1)^{k-in-1}\binom{k-2}{k-in-1}\frac{(S_{i+1}-S_i)}{mn}. \tag{6}$$

We now pass to b_k. As in the case of a_k, the difference $\Delta_h^k f(0)$ can be written as a linear combination of second differences, and once again many of these terms are zero because of the special nature of $f(x)$. Hence we are left with

$$\Delta_h^k f(0)=\sum_{i=1}^{l-1}(-1)^{k-in-1}\left[\binom{k-2}{k-in-1}\Delta_h^2 f\left(\frac{in-1}{mn+1}\right)-\binom{k-2}{k-in-2}\Delta_h^2 f\left(\frac{in}{mn+1}\right)\right]. \tag{7}$$

A calculation shows that

$$\Delta_h^2 f\left(\frac{in-1}{mn+1}\right) = \frac{(m-i)}{m(mn+1)}\,(S_{i+1} - S_i). \tag{8}$$

and

$$\Delta_h^2 f\left(\frac{in}{mn+1}\right) = \frac{i}{m(mn+1)}\,(S_{i+1} - S_i). \tag{9}$$

From (7), (8) and (9) we obtain

$$\Delta_h^k f(0) = \sum_{i=1}^{l-1} (-1)^{k-in-1}\left[\binom{k-2}{k-in-1}(m-i) - \binom{k-2}{k-in-2}\, i\right] \times \frac{(S_{i+1} - S_i)}{m(mn+1)}. \tag{10}$$

Another calculation yields

$$\binom{k-2}{k-in-1}(m-i) - \binom{k-2}{k-in-2}\, i = \binom{k-2}{k-in-1}\left(\frac{mn+1-k}{n}\right). \tag{11}$$

so that, from (10) and (11), we have

$$b_k = \binom{mn+1}{k}\sum_{i=1}^{l-1} (-1)^{k-in-1}\binom{k-2}{k-in-1}\frac{(mn+1-k)}{mn(mn+1)}\,(S_{i+1} - S_i). \tag{12}$$

Now $\binom{mn+1}{k}(mn+1-k)/(mn+1) = \binom{mn}{k}$, so that $b_k = a_k$. Note, also, from (12), that if $k = mn+1$, then $b_k = 0$.

Case 2. Suppose $k = (l-1)n + 1$ for some $l = 1, 2, \ldots, m$.

In this case a_k is unchanged from Case 1, but in b_k the term involving $\Delta_h^2 f((l-1)n/(mn+1))$ is missing, with the remaining terms unchanged. So the only term that must be investigated is that involving $S_l - S_{l-1}$. Here we find that the coefficient of this term in a_k is $\binom{mn}{k}(1/mn)$, while in b_k it is

$$\begin{aligned}\binom{mn+1}{k}\frac{(m-(l-1))}{m(mn+1)} &= \frac{(mn+1)!}{k!(mn+1-k)!}\,\frac{(m-(l-1))}{m(mn+1)} \\ &= \frac{(mn)!}{k!(mn+1-k)!}\,\frac{(m-(l-1))}{m} \\ &= \binom{mn}{k}\frac{(m-(l-1))}{m(mn+1-k)}\end{aligned}$$

$$= \binom{mn}{k} \frac{1}{mn} \left[\frac{(m - (l-1)n}{mn + 1 - k} \right]$$

$$= \binom{mn}{k} \frac{1}{mn} \left[\frac{mn - (l-1)n}{mn + 1 - k} \right]$$

$$= \binom{mn}{k} \frac{1}{mn} \left[\frac{mn - (l-1)n}{mn - (l-1)n} \right] = \binom{mn}{k} \frac{1}{mn}.$$

Therefore, $b_k = a_k$ in this case as well.

The only remaining case is $k = 0$, which is easily disposed of since $b_0 = a_0 = f(0)$. The proof is now complete.

REFERENCE

1. G. G. LORENTZ, "Bernstein Polynomials," Univ. of Toronto Press, Toronto, 1953.